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A Generalization of the One-Dimensional δ-Function Bose Gas

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The one-dimensional boson gas with δ-function interaction is modified to include arbitrary pseudopotential interaction. The system is shown to be solvable by the Bethe Ansatz for certain classes of pseudopotential.

KEY WORDS: Bose gas; one-dimensional fluid; point interaction; pseudopotential; Bethe Ansatz.

1. INTRODUCTION

In the theory of Bose fluids, exactly soluble model systems are unfortunately rare. Of the few many-body Hamiltonians which allow exact solution, the one-dimensional Bose gas with δ-function pair interaction has perhaps the most realistic pair potential. In the case of impenetrable point particles, Girardeau[(1)] showed that the energy spectrum was identical to that of the free Fermi gas. Subsequently, Lieb and Liniger[(2)] extended the solution to include δ-function pair interaction, for any fixed, positive δ-function strength. This resulted in a one-parameter family of solutions, the extremes of which were, respectively, the impenetrable case of Girardeau and the completely free-particle case. Lieb and Liniger[(3)] further showed that there appeared to be two excitation spectra associated with the gas, an interesting and rather unexpected result. The $T>0$ case was subsequently analyzed by Yang and Yang,[(5)] who also showed the completeness of the states obtained by the Bethe Ansatz.[(4)] Several reviews of various aspects of the Bethe Ansatz along with extensive bibliographies may be found in ref. 8.

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In this paper we further extend the boson model to include arbitrary pseudopotential interaction. That is, we shall show that the Bethe Ansatz yields a solution to the model equation

$$\left(-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+2\sum_{i>j}\sum_{n}c_n\left[\delta(x_i-x_j)\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)^{2n}\right]\right)\Psi(x_1,..., x_N)$$
$$=E\Psi(x_1,..., x_N) \tag{1}$$

Here the pair potential may be regarded as a form of pseudopotential, where the even derivatives in the potential act to "tune" the effective delta-function strength according to the relative "velocities" of the interacting particles. The above "potential" as it stands is not Hermitian, or even well defined. It is to be interpreted by carrying out the derivatives to the right of $x_i = x_j$. From the symmetry of the Bose wave function, this is identical with averaging the derivatives to the right and to the left, and then indeed the potential becomes Hermitian.

A motivation for considering (1) is that we may choose the coefficients of the pseudopotential to imitate real finite-range potentials. In such cases Eq. (1) would be valid as a dilute-limit approximation to the actual potential.

In the next section we state the Lieb result for bosons with fixed δ-function strength, and then interpret this from a scattering viewpoint. This will allow us to write down the form of the solutions to (1) immediately. In Section 3 we solve explicitly a restriction of (1), to verify the simple picture invoked in the second section. In Section 4 we show that the Bethe Ansatz works in general for Eq. (1), and in Section 5 we discuss conditions necessary for the completeness of states.

2. THE SCATTERING PICTURE

The ground-state energy of a system of N bosons with repulsive δ-function interaction in one dimension was calculated by Lieb and Liniger.[2] The Hamiltonian for the system is, for $c > 0$,

$$H = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+c\sum_{i>j}\delta(x_i-x_j) \tag{2}$$

where the units are chosen so that $\hbar = 2m = 1$. Using Bethe's hypothesis,[4] Lieb and Liniger showed that the wave vectors k of the hypothesis must satisfy

$$(-1)^{(N-1)}\exp(-ikL)=\exp\left[i\sum_{k'}\theta(k'-k)\right] \tag{3}$$

where, for $-\pi \leqslant \theta \leqslant \pi$,

$$\theta(k) = -2 \arctan(k/c) \tag{4}$$

Here the length of the periodic box is L.

Equation (3) is obtained by imposing periodic boundary conditions on the N-particle system, which itself has a large set of "internal boundaries" corresponding to hyperplanes of particle pair interaction. At first sight it appears somewhat miraculous that the internal boundary conditions may be satisfied by a wavefunction of the Bethe form [see Eq. (27)]. That is, one would normally expect diffraction to occur. (McGuire[6,7] has examined the three-particle case in detail and has established general conditions for the appearence of diffraction. The equal mass fixed-strength delta-function case was shown to be diffractionless.) However, the simplicity of the final result (3) for the N-particle case suggests that there is, perhaps, a simple mechanism at the root of the problem which may be amenable to generalization. In this section we shall, for the time being, set aside details, and propose a simple picture to motivate an extension of Eq. (3) to other potentials.

Let us consider the simple system of a single particle in a periodic box of length L, with a single delta-function scattering center at $x=0$. The stationary solutions will be superpositions of left and right incident waves ψ_- and ψ_+, respectively, with

$$\psi_\pm = \begin{cases} e^{\mp ikx} + re^{\pm ikx}, & \pm x \geqslant 0 \\ te^{\mp ikx}, & \pm x < 0 \end{cases} \tag{5}$$

where r and t are the reflection and transmission coefficients, respectively. The continuity of ψ at $x=0$, periodicity, and the condition $\psi'(0^+) - \psi'(0^-) = 2c\psi(0)$ require that the left and right incident waves have equal amplitude and that

$$r = \frac{c}{ik - c} \tag{6}$$

$$t = \frac{ik}{ik - c} \tag{7}$$

or

$$r + t = -\exp(i\theta_k) \tag{8}$$

where

$$\theta_k = -2 \arctan(k/c)$$

The scattering "picture" we use is as follows. Stationary solutions of the particle in a box consist of pairs of waves moving in opposite directions. Each wave experiences a phase shift $\theta_k + \pi$ when passing the scattering center at $x=0$. Thus, for example, the periodic boundary condition $e^{ikL}=1$ for k would be modified by the presence of a scatterer to read

$$e^{i(kL+\theta_k)} = -1$$

Notice that this would also be the case if $\psi(x)$ represented the wavefunction of the relative coordinates of two identical bosons.

Let us carry this "picture" over to the case of $N=2n$ point bosons in a box. We suppose that there are N waves present with wave numbers $\{k_i, i=1,\dots, N\}$. In the absence of interaction, the k_j satisfy the periodic boundary conditions $e^{ik_jL}=1$. However, with interaction each wave k_j feels the presence of $(2n-1)$ scattering centers with relative wavenumbers $\{k_j-k_i\}$, and the boundary condition for k_j becomes simply

$$-\exp(-ik_jL) = \exp\left[i\sum_{k_i} \theta(k_i-k_j)\right] \tag{9}$$

This is just the relation (3), for even N. [For odd N the wave k_j scatters off an even number of particles in one period and the factors of -1 in Eq. (8) cancel, removing the minus sign from (9).]

The simplicity of the above picture lies in the fact that each "wave" k_j treats every other wave k_i as an independent scattering center with relative wavenumber (k_i-k_j). Neither the position nor the relative ordering of the particles is relevant to this discription. This suggests that if we keep point particle interaction, with a potential strength dependent only on the relative velocities of the particles, Eq. (9) should remain correct with a suitable redifinition of θ.

This is in fact the case. However, to illustrate some of the details involved, we consider the example of a three-particle system with a second-order potential.

3. A THREE-PARTICLE PROBLEM

The heuristic picture sketched above indicates that if the Bethe Ansatz works for a potential, then the distribution of k's will be determined by the phase relation (3). On the other hand, the picture also suggests that the Ansatz should work, provided that the scattering conditions apply only on the interaction hyperplanes, and that they depend only on relative wavenumber. To see that this is actually the case, it is helpful to solve a simple system in detail. We shall consider the Hamiltonian

$$H = -\sum_{i=1}^{3} \frac{\partial^2}{\partial^2 x_i} + c[\delta(x_1 - x_2) + \delta(x_2 - x_3) + \delta(x_3 - x_1)] + d \sum_{i<j}^{3} \delta(x_i - x_j)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2 \qquad (10)$$

The boundary conditions resulting from integrating across the hyperplane $x_j = x_i$ are then

$$\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i}\right)\psi\Bigg|_{x_j = x_i^-}^{x_j = x_i^+} = \left[c + d\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right]\psi\Bigg|_{x_j = x_i^+} \qquad (11)$$

We now assume a solution in R_1: $0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant L$ of the form

$$\begin{aligned}\psi &= e^{i(k_1x_1 + k_2x_2 + k_3x_3)} + a(213)\, e^{i(k_2x_1 + k_1x_2 + k_3x_3)} \\ &\quad + a(231)\, e^{i(k_2x_1 + k_3x_2 + k_1x_3)} \\ &\quad + a(321)\, e^{i(k_3x_1 + k_2x_2 + k_1x_3)} + a(312)\, e^{i(k_3x_1 + k_1x_2 + k_2x_3)} \\ &\quad + a(132)\, e^{i(k_1x_1 + k_3x_2 + k_2x_3)} \\ &\equiv e_1 + a(213)\, e_2 + a(231)\, e_3 + a(321)\, e_4 + a(312)\, e_5 + a(132)\, e_6 \end{aligned} \qquad (12)$$

Here we have written the coefficients according to the permutation of the k's in the accompanying exponentials, and shortened the notation in an obvious way. We now further assume that the coefficients may be expressed as products of nearest neighbor interchange coefficients, where nearest neighbor exchange corresponds to simply transposing a single adjacent pair of k's. That is, we suppose that we may write

$$\begin{aligned} a(213) &= a(21) \\ a(231) &= a(21)\, a(31) \\ a(321) &= a(21)\, a(31)\, a(32) \\ a(312) &= a(21)\, a(31)\, a(32)\, a(12) \\ a(132) &= a(21)\, a(31)\, a(32)\, a(12)\, a(13) \end{aligned} \qquad (13)$$

If we require that $a(ij)\, a(ji) = 1$, our assumed form for ψ is

$$\begin{aligned}\psi(x_1, x_2, x_3) &= e_1 + a(21)\, e_2 + a(21)\, a(31)\, e_3 \\ &\quad + a(21)\, a(31)\, a(32)\, e_4 + a(31)\, a(32)\, e_5 + a(32)\, e_6 \end{aligned} \qquad (14)$$

We now apply the boundary condition (11) at $x_1 = x_2 = x$. Using the notation

$$e_{1.} = e^{i[(k_1+k_2)x + k_3 x_3]}, \qquad e_{3.} = e^{i[(k_2+k_3)x + k_1 x_3]}$$
$$e_{5.} = e^{i[(k_1+k_3)x + k_2 x_3]} \tag{15}$$

we calculate and collect derivatives. The left-hand side of (11) becomes

$$i\{(k_2-k_1)[1-a(21)]\, e_{1.} + a(21)\, a(31)(k_3-k_2)[1-a(32)]\, e_{3.} + a(32)(k_3-k_1)[1-a(31)]\, e_{5.}\}$$

The right-hand side is

$$[c-d(k_2-k_1)^2][1+a(21)]\, e_{1.} + a(21)\, a(31)[c-d(k_3-k_2)^2]\, e_{3.} + a(32)[c-d(k_3-k_1)^2][1+a(31)]\, e_{5.}$$

Since the $e_{i.}$ are linearly independent, we must have, for example,

$$i(k_2-k_1)[1-a(21)] = [c-d(k_2-k_1)^2][1+a(21)]$$

or

$$a(21) = \frac{i(k_2-k_1) - [c-d(k_2-k_1)^2]}{i(k_2-k_1) + [c-d(k_2-k_1)^2]} \tag{16}$$

The two remaining coefficients may be obtained similarly, and the result is

$$a(ij) = \frac{i(k_i-k_j) - [c-d(k_i-k_j)^2]}{i(k_i-k_j) + [c-d(k_i-k_j)^2]} \tag{17}$$

The other "internal" boundary in R_1, namely the hyperplane $x_2 = x_3$, yields the same coefficients.

Comparing Eq. (17) to the fixed-strength case, which was

$$a_0(ij) = -\frac{c - i(k_i-k_j)}{c + i(k_i-k_j)} \tag{18}$$

we see that a change from a potential $c\sum_{i>j}\delta(x_i-x_j)$ to a potential

$$c\sum_{i>j}\delta(x_i-x_j) + d\sum_{i>j}^{3}\delta(x_i-x_j)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2$$

simply changes the constant c in the phase factor to the k-dependent "constant" $c \to c - dk^2$.

So far we have verified that the solution assumed in (16) does in fact conform to the boundary condition (11) provided that the $a(ij)$ are given by (17). That is, we have found a functional form for ψ that will satisfy the internal boundary conditions for any nondegenerate set of k's.

The "external" boundary conditions in R_1 must also be satisfied, i.e.,

$$\psi(0, x_2, x_3) = \psi(x_2, x_3, L) \tag{19}$$

$$\frac{\partial}{\partial x_1} \psi(x_1, x_2, x_3)\bigg|_{x_1=0^+} = \frac{\partial}{\partial x_1} \psi(x_2, x_3, x_1)\bigg|_{x_1=L^-} \tag{20}$$

Substituting (16) into the boundary condition (19) and equating coefficients, we find

$$\begin{aligned} e^{-ik_1L} &= a(21)\, a(31) \\ e^{-ik_2L} &= a(12)\, a(32) \\ e^{-ik_3L} &= a(13)\, a(23) \end{aligned} \tag{21}$$

Similarly, the second condition (20) generates the same relations.

Now Eq. (17) may be rewritten

$$a(ij) = -e^{i\theta_{ji}}$$

where

$$\theta_{ij} = -2 \arctan\left(\frac{k_i - k_j}{c - d(k_i - k_j)^2}\right) \tag{22}$$

and (22) becomes

$$\exp(-ik_iL) = \exp\left(\sum_j \theta_{ij}\right) \tag{23}$$

which is just (3), as expected.

Thus, the simple scattering picture holds for the three-particle case where the interparticle potential contains a second-order term.

4. THE N-PARTICLE CASE

Returning to the N-particle equation (1), we note that we may restrict our attention to the region $R_1 : 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$, the remaining regions being implicitly defined through Bose symmetry. Inside R_1, ψ

satisfies the free-particle equation. On the "internal" boundaries the potentials give rise to the boundary conditions

$$\left(\frac{\partial}{\partial x_{j+1}}-\frac{\partial}{\partial x_j}\right)-\sum_n c_n\left(\frac{\partial}{\partial x_{j+1}}-\frac{\partial}{\partial x_j}\right)^{2n}\psi(x_1, x_2,..., x_N)\bigg|_{x_{j+1}=x_j}=0 \tag{24}$$

The periodic boundary conditions may be expressed in R_1 as

$$\psi(0, x_2,..., x_N)=\psi(x_2,..., x_N, L) \tag{25}$$

and

$$\frac{\partial}{\partial x}\psi(x_1, x_2,..., x_N)\bigg|_{x=0}=\frac{\partial}{\partial x}\psi(x_1,..., x_N, x)\bigg|_{x=L} \tag{26}$$

We now make the Ansatz that, for some ordered set $I=\{k_1, k_2,..., k_N\}$ of distinct wavenumbers, the wavefunction is given by

$$\psi(x_1, x_2,..., x_N)=\sum_P A(P)\exp\left[i\sum_{j=1}^{N} k_{P(j)}x_j\right] \tag{27}$$

where $\sum_P$ sums over all permutations of the set of wavenumbers, $A(P)$ is the amplitude associated with the permutation P, and the argument of the exponential is a sum over the permuted set of wavenumbers.

Now the internal boundary conditions (24) relate the components of ψ through adjacent transpositions of the sets P. That is, suppose

$$P=\{k_{\alpha_1},..., p, q,..., k_{\alpha_N}\}$$

and

$$Q=\{k_{\alpha_1},..., q, p,..., k_{\alpha_N}\}$$

differ only through the transposition of adjacent elements p and q at positions j and $j+1$, respectively. Then Eq. (24) requires that

$$\begin{aligned} i(q-p)[A(P)-A(Q)]\exp\left[i(p+q)x+\sum_i k_{\alpha_i}x_i\right] \\ =\sum_n c_n(-1)^n(q-p)^{2n}[A(P)+A(Q)] \\ \times\exp\left[i(p+q)x+\sum_i k_{\alpha_i}x_i\right] \end{aligned} \tag{28}$$

where the sum on the exponentials is over all $k_{\alpha_i} \neq q, p$. Canceling the exponentials, we are left with

$$\frac{A(Q)}{A(P)} \equiv a(p, q) = \frac{\sum (-1)^n c_n (p-q)^{2n} - i(q-p)}{\sum (-1)^n c_n (p-q)^{2n} + i(q-p)} \tag{29}$$

Now the above equation relates only adjacent transpositions such as P and Q. If we define

$$A(I) = 1 \tag{30}$$

where I is the set of wavevectors in its natural order, we may clearly define $A(P)$ to be the product of amplitudes along a path of adjacent transpositions from I to P. That is, if (p_1, q_1), (p_2, q_2),..., (p_n, q_n) is a sequence of adjacent transpositions taking I to P, then we may define A to be

$$A(P) = a(p_1, q_1)\, a(p_2, q_2) \cdots a(p_n, q_n)$$

However, there are many paths in permutation space from I to P and we must ensure that the $A(P)$ are unique. To see that they are in fact unique, suppose that P_1 and P_2 are two different "paths" from I to P. If $A(P_1)$ and $A(P_2)$ are the respective amplitudes, then since

$$P_1 = P_1(P_2^{-1} P_2) = (P_1 P_2^{-1})\, P_2$$

we must have

$$A(P_1 P_2^{-1}) = A(P_1)\, A(P_2^{-1}) = 1$$

That is, for $A(P)$ to be uniquely defined by (29) and (30), $A(L)$ must be unity for all closed paths L which take a permutation into itself. For example, Fig. 1 shows an adjacent transposition loop for a set of four k's. Tracing the trajectory of any k in such a loop, it is apparent that if the path for k_1, say, crosses the path for k_2, then it will do so an even number of times, alternating in its direction of approach. Each such intersection will contribute a factor of $a(k_1, k_2)$ or $a(k_2, k_1)$, respectively, corresponding to an approach from the left or the right. Since the intersections are paired, the final product is necessarily unity provided that

$$a(p, q)\, a(q, p) = 1$$

However, this is clearly the case, as may be seen from the definition (29). Thus, provided that the k's are distinct, the assumed solution (27) satisfies the internal boundary conditions, provided that the amplitudes are chosen according to Eq. (29).

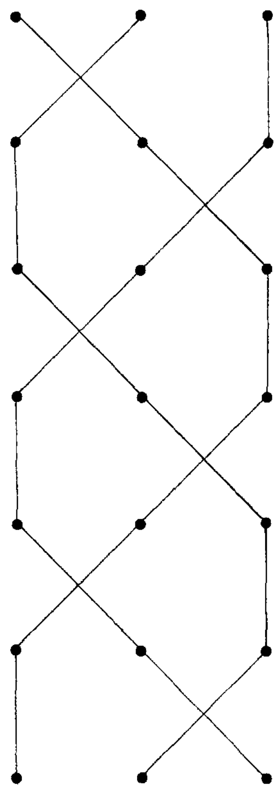

Fig. 1. A closed loop of adjacent transpositions.

The effect of the periodic boundary conditions (26) may be obtained by substituting (27) into (26), which results in the requirement

$$(-1)^{N-1} e^{-ik_jL} = \prod_{n \neq j} a(k_j k_n) \tag{31}$$

If we write

$$a(p, q) = -\exp[-i\tilde{\theta}(p-q)] \tag{32}$$

where

$$\tilde{\theta}(p-q) = -2 \arctan\left[(p-q)\Big/\sum_n (-1)^n c_n (p-q)^{2n}\right] \tag{33}$$

for

$$-\pi \leqslant \tilde{\theta} < \pi$$

then Eq. (31) becomes

$$(-1)^{N-1}\exp(-ik_jL)=\exp\left[\sum_n \tilde{\theta}(k_n-k_j)\right] \tag{34}$$

This is precisely the relation (31) of Lieb and Liniger,[2] where, in this context, $\tilde{\theta}$ is given by the function (33).

Thus the Bethe Ansatz does provide a solution to the Hamiltonian of Eq. (1). In the following section we shall discuss conditions under which the Ansatz provides a basis for all states.

5. THE BASIS OF STATES

Having established that the Bethe hypothesis provides solutions to the model equations, we would like to find conditions under which all solutions are found this way. Although the following argument is not rigorous, it is suggestive that a large class of pseudopotentials are completely covered by the Ansatz.

Following the work of Yang and Yang,[5] we take the logarithm of (33) to find

$$k_jL=2\pi I_j+\sum_{k'}\tilde{\theta}(k_j-k') \tag{35}$$

where I_j is integer or half integer, respectively, for odd or even N. Writing

$$\tilde{\theta}(k)=\int_0^k \tilde{\theta}(x)\,dx \tag{36}$$

and defining

$$B(k_1,..., k_N)=\frac{1}{2}L\sum_1^N k_j^2-2\pi\sum_1^N I_jk_j-\frac{1}{2}\sum_{j,l}\tilde{\theta}(k_j-k_l) \tag{37}$$

we note that Eq. (35) is the extremum condition for (37). However, the second derivative matrix of B is

$$\frac{\partial^2 B}{\partial k_j\,\partial k_l}=\delta_{jl}\left[L-\sum_s \tilde{\theta}'(k_j-k_s)\right]+\tilde{\theta}'(k_j-k_l) \tag{38}$$

Let us suppose that $-\tilde{\theta}'$ is nonnegative. The second derivative matrix B'' is then positive definite. Thus, since for large k, $B\sim L\sum k_i^2$, then for any set of numbers $\{I_i\}$, the extremum given by (35) is a minimum, and is unique. Now if, in the general potential of Eq. (1) we set $c_0^{-1}=0$, then we have the impenetrable particle case. Since dk/dc^{-1} is continuous for $c > 0$, provided that when we move c_0^{-1} to a finite, positive target value, the positivity of B'' is not violated, then, as in the fixed delta-function case, we expect the completeness of the set of Bethe eigenfunctions to follow.

Now, nonnegativity of $\tilde{\theta}'$ places a restriction on the class of pseudopotentials for which we can be reasonably certain of the completeness of states. However, even with this restriction we may investigate some new qualitative features. For example, if we consider the pseudopotential of Eq. (1) with c_0, $c_1 > 0$ and $c_n = 0$, $n > 2$, we see that the net effect at high k is that of attraction. However, we have already seen from the three-particle case that $\tilde{\theta}$ is given by Eq. (22), so that

$$\tilde{\theta}'(k) = -2 \frac{c_0 + 3dk^2}{k^2 + (c_0 - dk^2)^2} \tag{39}$$

is nonpositive and hence completely amenable to the Bethe Ansatz. In a subsequent publication we shall investigate numerically the effects of such attractions on the ground state and excitations.

McGuire (personal communication) has suggested that the nonpotential nature of (1) makes the completeness argument suspect; our conclusions as to completeness must be regarded as tentative.

6. DISCUSSION

We have introduced a simplified scattering picture of a one-dimensional Bose fluid in which the interparticle potential has zero range and is velocity dependent. This picture is not exactly new. McGuire[6] has used a similar approach involving ray tracing and an electromagnetic analogy. An even closer physical picture may be found in the work of Sutherland.[8] The primary novelty of our approach is the extension of the delta-function interaction to potentials with velocity-dependent strengths. This provides a new class of exactly solvable models which should serve as a good testing ground for approximation methods.

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